Metamorphoses of Manifold-Valued Images

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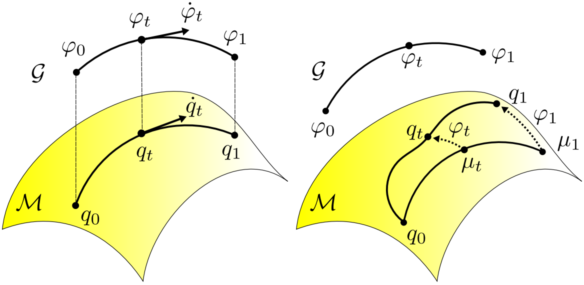
- **Metamorphosis** = pair of curves (φ_t, μ_t) in $\mathcal{G}$ and $\mathcal{M}$ inducing an image curve $q_t = \varphi_t(\mu_t)$ in $\mathcal{M}$ Trouvé and Younes [2005].

$$E(v,z) = \frac{1}{2}\int_0^1 |v_t|_V^2 \,\mathrm{d}t + \frac{1}{2\sigma^2}\int_0^1 |z_t|_{q_t}^2 \,\mathrm{d}t \longrightarrow \min \tag{1.1}$$

subject to $\dot{q}_t = v_t(q_t) + z_t$, $q_0 = q^{(0)}$ and $q_1 = q^{(1)}$.



LDDMM Metamorphoses

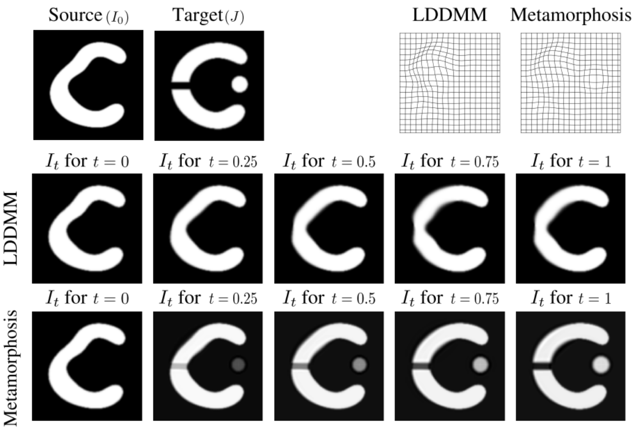


Fig. 2. Comparison of LDDMM vs Metamorphoses registration *Top-right* final deformation grids obtained by integrating over all vector fields $(v_t)_{t\in[0,1]}$. *Bottom Rows* Image evolution during the geodesic shootings of the respective method after optimisation with Eq. 4.

Figure – From François et al. [2021]

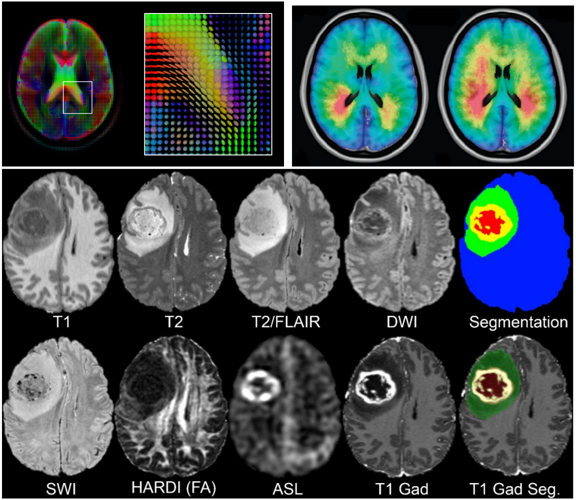


Figure – Examples of image modalities Ellingson et al. [2013]; Tournier [2019]; Calabrese et al. [2021].

- **Manifold-valued images** $= L^2(\Omega, M)$ where $\Omega \subset \mathbb{R}^d$ and M is a n-dimensional manifold equipped with a metric $\langle \cdot, \cdot \rangle_m$ at point $m \in M$.
- **Admissible deformations**: $G_V = \{\varphi_1 \mid v \in L^2([0,1], V)\}$ where φ_1 is the flow a time 1 resulting from the integration of the following ODE:

$$\begin{cases} \dot{\varphi}_t = v_t \circ \varphi_t \\ \varphi_0 = \mathrm{id} \end{cases}$$

- **Metamorphoses of manifold-valued images**:

$$E(v, z) = \frac{1}{2} \int_0^1 |v_t|_V^2 \, \mathrm{d}t + \frac{1}{2\sigma^2} \int_0^1 \int_\Omega |z_t|_{q_t}^2 \, \mathrm{d}x \, \mathrm{d}t \longrightarrow \min \tag{2.1}$$

subject to $\dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t$, $q_0 = q^{(0)}$ and $q_1 = q^{(1)}$.

- Optimal solutions of (2.1) verify

$$\begin{cases} \dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t \\ \dot{z}_t = -\mathrm{div}(z_t \otimes v_t) + \sigma(z_t) \\ v_t = -K_V(z_t^\flat(\mathrm{d}q_t)) \end{cases}$$

where $\sigma(z_t) \triangleq z_t^i\, z_t^j\, \Gamma_{ij}^k(q_t)\, \partial_{q_t^k}$ and $z_t^\flat(\mathrm{d}q_t) \triangleq g_{ij}(q_t)\, z_t^i\, \mathrm{d}_x q_t^j$.

Remark: If $v_t = 0$, $\dot{q}_t = z_t$ and the second equation yields

$$\ddot{q}_t^k + \dot{q}_t^i \dot{q}_t^j \Gamma_{ij}^k(q_t) = 0$$

- Introduce the control-dependent Hamiltonian associated with (2.1):

$$H(p,q,v,z) = \int_\Omega \left[(p| - \mathrm{d}q \cdot v + z) - \frac{1}{2}|z|_q^2 \right] \mathrm{d}x - \frac{1}{2}|v|_V^2$$

- Optimal solutions of (2.1) satisfy:

$$\begin{cases} \dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t \\ \dot{p}_t = -\mathrm{div}(p_t \otimes v_t) + \sigma^*(z_t) \\ v_t = -K_V(p_t(\mathrm{d}q_t)) \\ z_t = p_t^\sharp \end{cases}$$

where $\sigma^*(z_t) \triangleq z_t^i\, z_t^l\, \Gamma_{ij}^k(q_t)\, g_{kl}(q_t)\, \mathrm{d}q_t^j$ and $p_t(\mathrm{d}q_t) \triangleq p_t^i\, \mathrm{d}_x q_t^i$.

Remark: $\dot{p}_t = -\mathrm{div}(p_t \otimes v_t) + \sigma^*(z_t) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \dot{z}_t = -\mathrm{div}(z_t \otimes v_t) + \sigma(z_t)$

$$\tilde{E}(v,z) = \frac{1}{2}\int_0^1 |v_t|_V^2 \,\mathrm{d}t + \frac{1}{2\sigma^2}\int_0^1 \int_\Omega |z_t \circ \varphi_t|_{\mu_t}^2 |\mathrm{d}\varphi_t| \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \min \tag{2.2}$$

subject to $\dot{\varphi}_t = v_t \circ \varphi_t$, $\dot{\mu}_t = z_t \circ \varphi_t$, $\mu_0 = q^{(0)}$ and $\mu_1 = q^{(1)} \circ \varphi_1$.

- Fixing v and minimizing w.r.t. z yields that an optimal curve μ_t verifies:

$$\nabla_{\dot{\mu}_t}(\dot{\mu}_t |\mathrm{d}\varphi_t|) = 0$$

Remark: One can show that an optimal curve μ_t is the reparametrization of a geodesic joining μ_0 and μ_1 by the mapping $s(t) \triangleq \frac{\int_0^t |\mathrm{d}\varphi_u|^{-1}\mathrm{d}u}{\int_0^1 |\mathrm{d}\varphi_u|^{-1}\mathrm{d}u}$.

- Optimal solutions of (2.2) verify

$$\begin{cases} \dot{\varphi}_t = v_t \circ \varphi_t \\ \dot{\mu}_t = z_t \circ \varphi_t \\ \frac{\mathrm{d}}{\mathrm{d}t}(z_t \circ \varphi_t |\mathrm{d}\varphi_t|) = \sigma(z_t) \circ \varphi_t \\ \cancel{v_t = -K_V(z_t^\flat(\mathrm{d}q_t))} \end{cases}$$

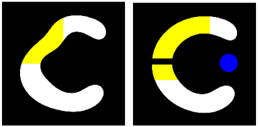
- Developing the second and third equations, we get:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(q_t \circ \varphi_t) = z_t \circ \varphi_t \\ \frac{\mathrm{d}}{\mathrm{d}t}(z_t \circ \varphi_t) = (-z_t \mathrm{div}(v_t) + \sigma(z_t)) \circ \varphi_t \end{cases}$$

- Semi-Lagrangian scheme (first-order approximation):

$$\begin{cases} q_{t+\delta t} = [q_t + \delta t\, z_t] \circ (\mathrm{id} - \delta t\, v_t) \\ z_{t+\delta t} = [z_t + \delta t(-z_t \mathrm{div}(v_t) + \sigma(z_t))] \circ (\mathrm{id} - \delta t\, v_t) \end{cases}$$

Remark: We can find a similar approximation by identifying the advection equation in the Eulerian form.



Source $q^{(0)}$ Target $q^{(1)}$

$$\begin{cases} \dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t \\ \dot{p}_t = -\mathrm{div}(p_t \otimes v_t) \\ v_t = -K_V(\mathrm{d}q_t^* p_t) \\ z_t = p_t \end{cases}$$

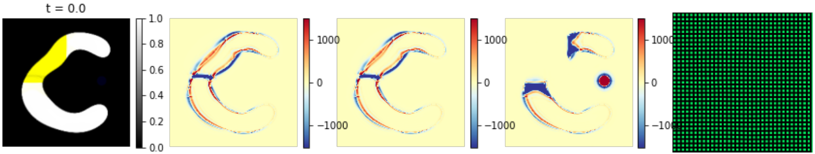
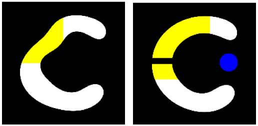


Figure – Toy example of a metamorphosis in $L^2(\Omega, \mathbb{R}^3)$



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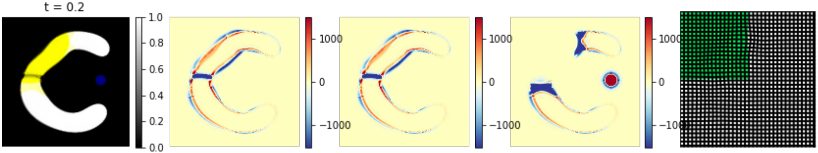

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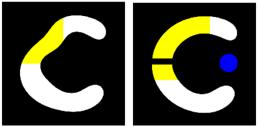

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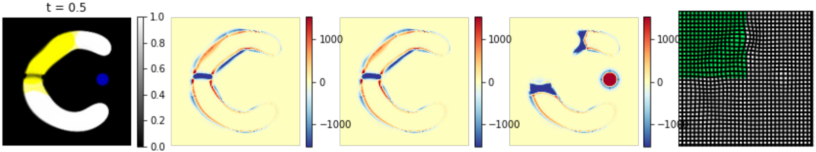

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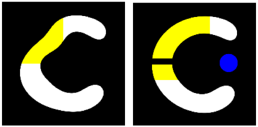

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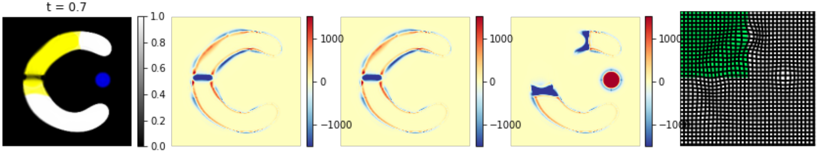

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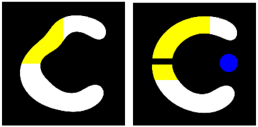

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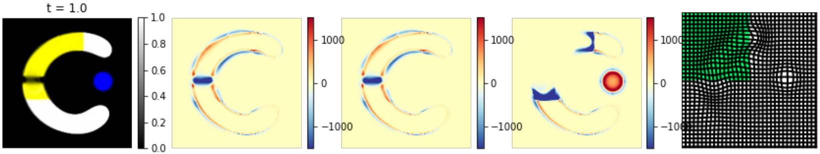

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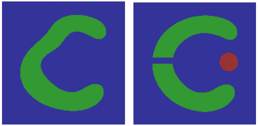

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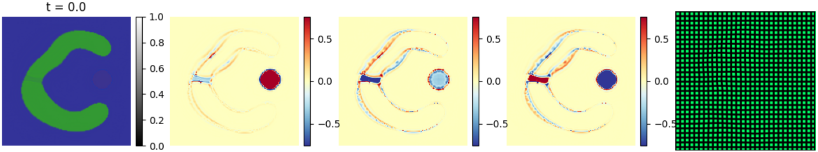
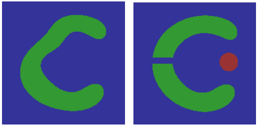


Figure – Toy example of a metamorphosis in $L^2(\Omega, \Delta^2)$.
Background: $(0.2, 0.2, 0.6)$, C shape: $(0.2, 0.6, 0.2)$, disk shape: $(0.6, 0.2, 0.2)$



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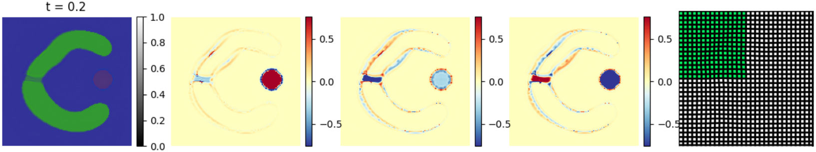

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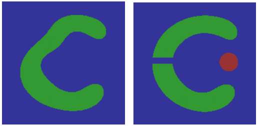

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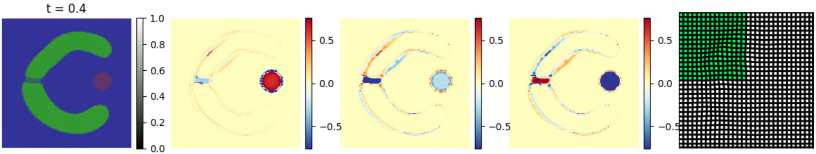

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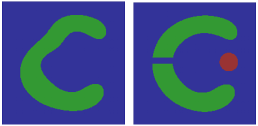

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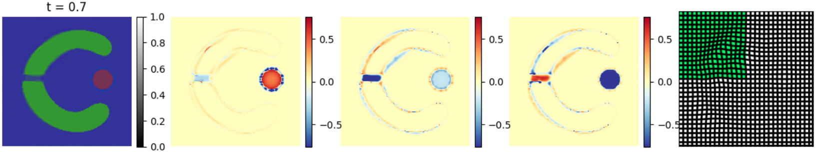

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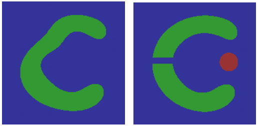

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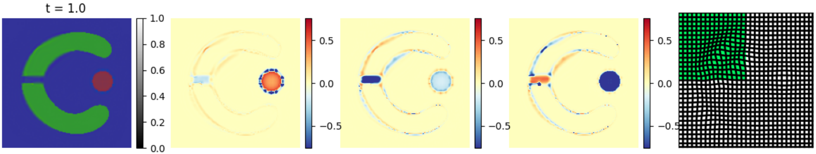

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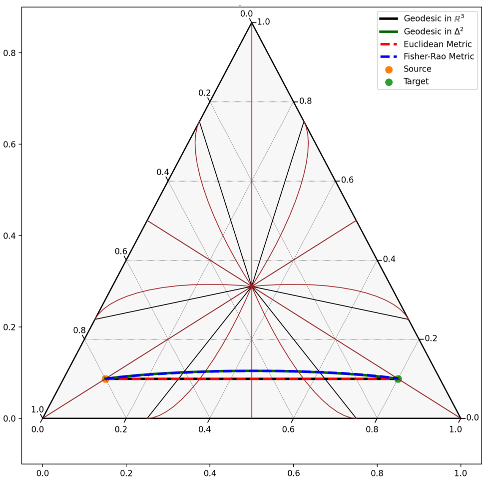
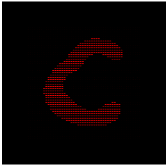


Figure – Pure metamorphosis in $L^2(\Omega, \Delta^2)$ (i.e. without deformation).
Source: $(0.1, 0.1, 0.8)$, Target: $(0.8, 0.1, 0.1)$



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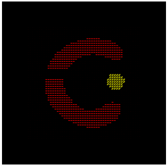

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$$

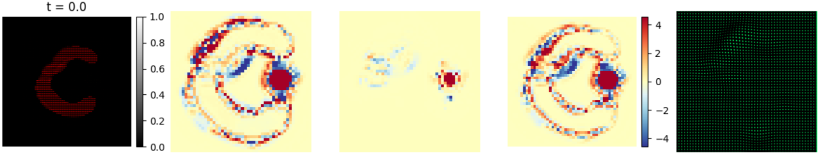

Figure – Toy example of a metamorphosis in $L^2(\Omega, \mathcal{S}^3_{++})$.

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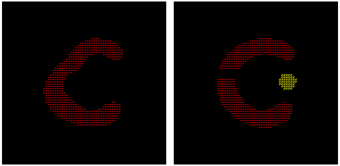

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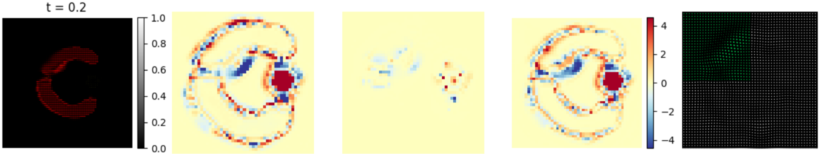

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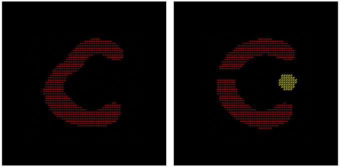

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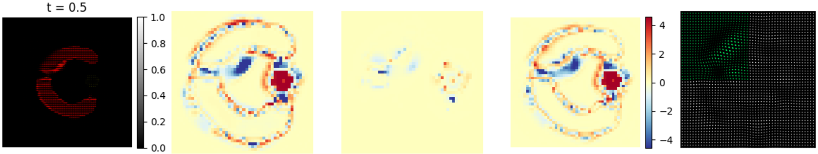

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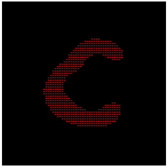

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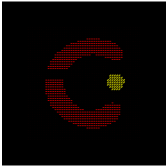


Target $q^{(1)}$

$$\begin{cases} \dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t \\ \dot{p}_t = -\mathrm{div}(p_t \otimes v_t) - (q_t^{-1} z_t)^2 q_t^{-1} \\ v_t = -K_V(\mathrm{Tr}(p_t \mathrm{d}q_t)) \\ z_t = q_t p_t q_t \end{cases}$$

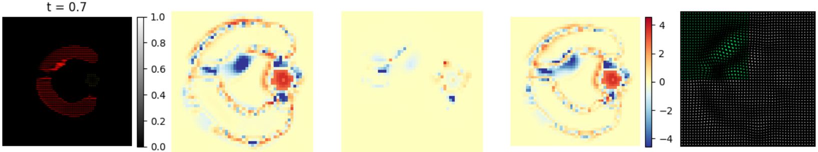
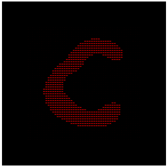
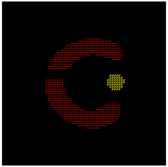


Figure – Toy example of a metamorphosis in $L^2(\Omega, \mathcal{S}^3_{++})$.

Background: $\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$, C shape: $\begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, disk shape: $\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$



Source $q^{(0)}$



Target $q^{(1)}$

$$\begin{cases} \dot{q}_t = -\mathrm{d}q_t \cdot v_t + z_t \\ \dot{p}_t = -\mathrm{div}(p_t \otimes v_t) - (q_t^{-1} z_t)^2 q_t^{-1} \\ v_t = -K_V(\mathrm{Tr}(p_t \mathrm{d}q_t)) \\ z_t = q_t p_t q_t \end{cases}$$

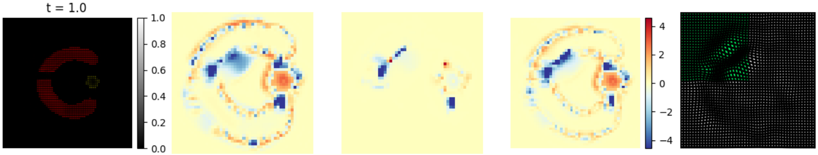


Figure – Toy example of a metamorphosis in $L^2(\Omega, \mathcal{S}^3_{++})$.
Background: $\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$, C shape: $\begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, disk shape: $\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$

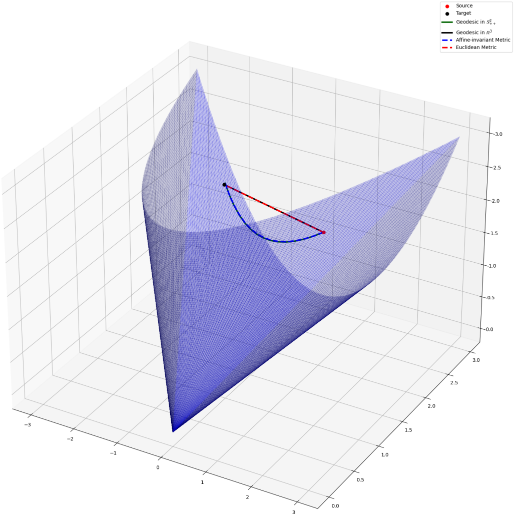


Figure – Pure metamorphosis in $L^2(\Omega, \mathcal{S}^3_{++})$ (i.e. without deformation).

Source: $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, Target: $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Contributions

- Existence and characterization of solutions for Metamorphoses of manifold-valued images
- Three examples: $\mathbb{R}^n$, Δ^n and $\mathcal{S}^n_{++}$
- Experiments on toy examples

PhD project (+ Pietro Gori & Jean Feydy)

- Working on real data: Probabilistic atlas of brain tumours
- Interactions with Optimal Transport (OT): data fidelity term, unbalanced OT, etc.

Thank you for your attention!

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