Extended orbit model

Thomas Pierron

ENS - centre Borelli

2023

Intro

- Molecular Computational Anatomy : Unifying the Particle to Tissue Continuum via Measure Representations of the Brain, Miller, Tward, Trouvé

Table of contents

Multi-scale LLDDMM

Shape space

$\boldsymbol{\mathcal{Q}} = \prod_{0 \leq l \leq L} \mathcal{Q}^l$ where $\mathcal{Q}^l$ space of landmarks, varifolds, etc...

Hierarchy of groups

We introduce extended group :

$$\mathbf{G}^k = \prod_{0 \leq l < L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

with component-wise law $\boldsymbol{\varphi} \cdot \boldsymbol{\varphi}' = (\varphi \circ \varphi')_{l \geq 0}$ and with component-wise action on $\boldsymbol{\mathcal{Q}}$: $\boldsymbol{\varphi} \cdot \mathbf{q} = (\varphi \cdot q)_{l \geq 0}$

RKHS

We consider a sequence of continuous embeddings $V_0 \hookrightarrow \cdots V_{L-1} \hookrightarrow V_L \doteq C_0^m(\mathbb{R}^d)$ and $\mathbf{V} = \prod_{0 \leq l < L} V_l$ equipped with the Hilbert norm defined by $|\mathbf{v}|^2_{\mathbf{V}} = \sum_{l=0}^{L-1} |v_l|^2_{V_l}$ for $\mathbf{v} = (u_l)_{\leq l < l} \in \mathbf{V}$.

Multi-scale LLDDMM

Shape space

$\boldsymbol{\mathcal{Q}} = \prod_{0\leq l\leq L} \mathcal{Q}^l$ where $\mathcal{Q}^l$ space of landmarks, varifolds, etc...

Hierarchy of groups

We introduce extended group :

$$\mathbf{G}^k = \prod_{0\leq l<L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

with component-wise law $\boldsymbol{\varphi} \cdot \boldsymbol{\varphi'} = (\varphi \circ \varphi')_{l\geq 0}$ and with component-wise action on $\boldsymbol{\mathcal{Q}}$: $\boldsymbol{\varphi} \cdot \mathbf{q} = (\varphi \cdot q)_{l\geq 0}$

RKHS

We consider a sequence of continuous embeddings $V_0 \hookrightarrow \cdots V_{L-1} \hookrightarrow V_L \doteq C_0^m(\mathbb{R}^d)$ and $\mathbf{V} = \prod_{0\leq l<L} V_l$ equipped with the Hilbert norm defined by $|\mathbf{v}|^2_{\mathbf{V}} = \sum_{l=0}^{L-1} |v_l|^2_{V_l}$ for $\mathbf{v} = (u_l)_{\leq l<l} \in \mathbf{V}$.

Multi-scale LLDDMM

Shape space

$\mathcal{\boldsymbol{Q}} = \prod_{0\leq l\leq L} \mathcal{Q}^l$ where $\mathcal{Q}^l$ space of landmarks, varifolds, etc...

Hierarchy of groups

We introduce extended group :

$$\mathbf{G}^k = \prod_{0\leq l<L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

with component-wise law $\boldsymbol{\varphi}\cdot\boldsymbol{\varphi'} = (\varphi\circ\varphi')_{l\geq 0}$ and with component-wise action on $\mathcal{\boldsymbol{Q}}$: $\boldsymbol{\varphi}\cdot\mathbf{q} = (\varphi\cdot q)_{l\geq 0}$

RKHS

We consider a sequence of continuous embeddings $V_0 \hookrightarrow \cdots V_{L-1} \hookrightarrow V_L \doteq C_0^m(\mathbb{R}^d)$ and $\mathbf{V} = \prod_{0\leq l<L} V_l$ equipped with the Hilbert norm defined by $|\mathbf{v}|^2_{\mathbf{V}} = \sum_{l=0}^{L-1} |v_l|^2_{V_l}$ for $\mathbf{v} = (u_l)_{\leq l<l} \in \mathbf{V}$.

Variational problem

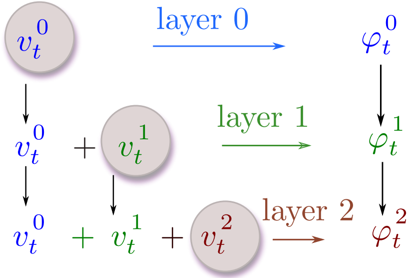

Figure – Hierarchical system of controls

$d_{\mathbf{G}^k}(\mathbf{Id}, \boldsymbol{\varphi}) = \inf_{\mathbf{u}, \dot{\boldsymbol{\varphi}} = \mathbf{u}_t \circ \boldsymbol{\varphi}} \frac{1}{2} \int_0^1 |\mathbf{A u_t}|^2 \, dt$ with constraints
$\boldsymbol{\varphi}(0) = \mathbf{Id}, \boldsymbol{\varphi}(1) = \boldsymbol{\varphi}$

Table of contents

Extended framework

We introduce a family of groups $\{G^l, l \in \mathbb{N}\}$ with following conditions

(G.1) G^{k+1} is a subgroup of G^k with smooth inclusion

(G.2) For $l > 0$, the inverse mapping on the restriction $\mathrm{inv} : G^{k+l} \to G^k$ is $\mathcal{C}^l$

(G.3) For $l > 0$, the induced multiplication

$$\begin{array}{ccc} G^{k+l} \times G^k & \longrightarrow & G^k \\ (g', g) & \longmapsto & g' \cdot g \end{array}$$

is $\mathcal{C}^l$, and is also $\mathcal{C}^\infty$ with regards to the first variable.

(G.4) For $l \geq 0$, the induced left infinitesimal action

$$\begin{array}{ccc} T_e G^{k+l} \times G^k & \longrightarrow & TG^k \\ (u, g) & \longmapsto & u \cdot g \end{array}$$

is a C^l mapping, and $\mathcal{C}^\infty$ with regards to the first variable.

Subriemannian geometry

Let $V \subset T_e G^{k+2}$ a Hilbert space. We define right-invariant metric on G^k induced by V.
Dynamic is given by

$$\begin{cases} g_0 = e \\ \dot{g}_t = u_t \cdot g_t = T_e R_{g_t}(u_t) \end{cases}$$

where $u_t \in V$. We define the energy of a path (g, u) as :

$$E(g, u) = \frac{1}{2} \int_I |u(t)|_V^2 dt$$

Proposition

We have global existence and uniqueness of normal geodesics

Subriemannian geometry

Let $V \subset T_e G^{k+2}$ a Hilbert space. We define right-invariant metric on G^k induced by V.
Dynamic is given by

$$\begin{cases} g_0 = e \\ \dot{g}_t = u_t \cdot g_t = T_e R_{g_t}(u_t) \end{cases}$$

where $u_t \in V$. We define the energy of a path (g, u) as :

$$E(g, u) = \frac{1}{2} \int_I |u(t)|_V^2 dt$$

Proposition

We have global existence and uniqueness of normal geodesics

Hamiltonian flow and Euler Poincaré equations

Hamiltonian

$$\mathcal{H}_{G^k} : \left| \begin{array}{ccc} TG^{k*} \times V & \longrightarrow & \mathbb{R} \\ (g,\ p,\ u) & \longmapsto & (p, u \cdot g) - \frac{1}{2}|u|_V^2 \end{array} \right.$$

Normal geodesics are solutions of the hamiltonian equations

$$\begin{cases} \dot{g} \ = \partial_p \mathcal{H}_{G^k}(g, p, u) \\ \dot{p} = -\partial_q \mathcal{H}_{G^k}(g, p, u) \\ \partial_u \mathcal{H}(g, p, u) = 0 \end{cases} \tag{1}$$

Hamiltonian flow and Euler Poincaré equations

The momentum $m_t = L_V u_t$ satisfies the equation :

$$\dot{m}_t + \mathrm{ad}^*_{u_t}(m_t) = 0 \tag{2}$$

or equivalently the integrated form :

$$\frac{d}{dt}\left(m_t \middle| \mathrm{Ad}_{g_t}(v)\right) = 0 \tag{3}$$

which leads to the equation for g :

$$\dot{g} = K\,\mathrm{Ad}^*_{g^{-1}}(m_0) \tag{4}$$

Metric on Banach manifold

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ Banach manifolds such that G^k acts on $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with some nice hypotheses. Let $(p_0^1, q_0^1, u_0^1) \in \mathcal{Q}_1 \times T\mathcal{Q}_1^* \times V$ and $(p_0^2, q_0^2, u_0^2) \in \mathcal{Q}_2 \times T\mathcal{Q}_2^* \times V$ such that $m_{\mathcal{Q}_1}(p_0^1, q_0^1) = m_{\mathcal{Q}_2}(p_0^2, q_0^2)$. Then for all $t \geq 0$, we have

$$m_{\mathcal{Q}_1}(p_t^1, q_t^1) = m_{\mathcal{Q}_2}(p_t^2, q_t^2) \text{ and } q_t^1 = g_t \cdot q_0^1, q_t^2 = g_t \cdot q_0^2$$

with g_t satisfying the equation :

$$\dot{g}_t = K m_{Q_i}(q_t, p_t) \cdot g_t$$

with $i \in \{0, 1\}$

Table of contents

Action on varifold

We consider the space of d-varifolds as the space of Radon Borel measures $\mathcal{M}(\mathbb{R}^d) = C_0(\mathbb{R}^d)'$.
For $\phi \in \mathrm{Diff}(\mathbb{R}^d)$, $\mu \in \mathcal{M}(\mathbb{R}^d)$, we define $\phi_* \mu$ as

$$\forall \omega \in C_0(\mathbb{R}^d),\ \phi_* \mu \cdot \omega = \mu \cdot \phi^* \omega$$

with $\phi^* \omega$ the pullback of ω by ϕ defined by the relation

$$\forall x \in \mathbb{R}^d,\ \phi^* \omega(x) = |J\phi(x)| \omega(\phi(x))$$

New action on varifolds

Instead consider semi-direct product :

$$H^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, \mathbb{R}^{+*}) \,.$$

where the group operation is given by $(\varphi, \alpha) \cdot (\varphi', \alpha') = (\varphi \circ \varphi', \alpha \circ \varphi' \alpha')$

$\mathrm{Diff}_{C_0^k}$ is embedded in H^k : $i_H : \varphi \mapsto (\varphi, |J_\varphi|)$ and thus V induces right-invariant metric on H^k

Action on varifolds :

$$\forall x \in \mathbb{R}^d, \; (\phi, \alpha)^* \omega(x) = \alpha \omega(\phi(x))$$

New action on varifolds

Instead consider semi-direct product :

$$H^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, \mathbb{R}^{+*})\,.$$

where the group operation is given by $(\varphi, \alpha) \cdot (\varphi', \alpha') = (\varphi \circ \varphi', \alpha \circ \varphi' \alpha')$

$\mathrm{Diff}_{C_0^k}$ is embedded in H^k : $i_H : \varphi \mapsto (\varphi, |J_\varphi|)$ and thus V induces right-invariant metric on H^k

Action on varifolds :

$$\forall x \in \mathbb{R}^d,\ (\phi, \alpha)^* \omega(x) = \alpha \omega(\phi(x))$$

New action on varifolds

Instead consider semi-direct product :

$$H^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, \mathbb{R}^{+*})\,.$$

where the group operation is given by $(\varphi, \alpha) \cdot (\varphi', \alpha') = (\varphi \circ \varphi', \alpha \circ \varphi' \alpha')$

$\mathrm{Diff}_{C_0^k}$ is embedded in H^k : $i_H : \varphi \mapsto (\varphi, |J_\varphi|)$ and thus V induces right-invariant metric on H^k

Action on varifolds :

$$\forall x \in \mathbb{R}^d,\ (\phi, \alpha)^* \omega(x) = \alpha \omega(\phi(x))$$

Co-adjoint variables

Let $\varphi_0 \in G^k$ and $p_0^H = (p_0^\varphi, p_0^\alpha) \in T^*_{i^H(\varphi_0)} H^k$ such that $p_0^G = (di_H)^* p_0^H$. Then for all time we have the following relation

$$p_t^G = (di_H)^* p_t^H \tag{5}$$

i.e.

$$p^G = p^\varphi - |J\varphi| \left(d\varphi^{-1}\right)^* \nabla p^\alpha \tag{6}$$

[1] Sylvain Arguillère. The general setting for shape analysis, 2015.

[2] Sylvain Arguillère, Emmanuel Trélat, Alain Trouvé, and Laurent Younes. Shape deformation analysis from the optimal control viewpoint. *Journal de Mathématiques Pures et Appliquées*, 104(1) :139–178, 2015.

[3] Sylvain Arguillère and Emmanuel Trélat. Sub-riemannian structures on groups of diffeomorphisms. *Journal of the Institute of Mathematics of Jussieu*, 16(4) :745–785, 2017.

[4] Sylvain Arguillere. Sub-riemannian geometry and geodesics in banach manifolds, 2020.

[5] Vladimir Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. In *Annales de l'institut Fourier*, volume 16, pages 319–361, 1966.

[6] Martin Bauer, Philipp Harms, and Peter W. Michor. Regularity and completeness of half-lie groups, 2023.

[7] Helge Glockner. Measurable regularity properties of infinite-dimensional lie groups, 2016.

[8] Andreas Kriegl and Peter W Michor. *The convenient setting of global analysis*, volume 53. American Mathematical Soc., 1997.

[9] S. Kurcyusz. On the existence and nonexistence of lagrange multipliers in banach spaces. *Journal of Optimization Theory and Applications*, 20 :81–110, 1976.

[10] S. Lang. *Fundamentals of Differential Geometry*. Graduate Texts in Mathematics. Springer New York, 2001.

[11] Hideki Omori and American Mathematical Society. *Infinite-dimensional Lie Groups*. Translations of mathematical monographs. American Mathematical Society, 1997.

[12] Peter W Michor and David Mumford. A zoo of diffeomorphism groups on $\mathbb{R}^n$. *Annals of Global Analysis and Geometry*, 44(4) :529–540, 2013.

[13] David Mumford and Peter W Michor. On euler's equation and epdiff. *arXiv preprint arXiv :1209.6576*, 2012.

[14] L. Younes. *Shapes and diffeomorphisms*. Springer, 2019.

[15] Michael Miller, Daniel Tward, and Alain Trouvé. Molecular computational anatomy : Unifying the particle to tissue continuum via measure representations of the brain. *BME Frontiers*, 2022, 2022.